- $cd_{\mathbb{Z}}(G) \le n$

$\Leftrightarrow \exists \cdots \to 0 \to 0 \to P_n \to \cdots \to P_0 \to \ell^\infty(X) \to 0$

a projective resolution

- $cd_{\mathbb{Z}}(G) \neq \sup\{cd_{\mathbb{Z}}$ of components of $G\}$.

<u>e.g.</u> $(X, G) = \bigsqcup_{i=1}^{\infty} (X_i, G_i)$

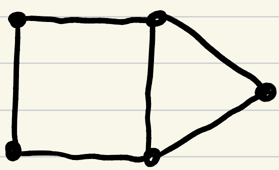
- G_i is L_i-Lipschitz equiv. to a tree.
- $L_i \to \infty$.

Example of projective resolution :

$$G = \coprod_{i=1}^{n} (\mathrm{graph}\,\varphi_i \sqcup \mathrm{graph}\,\varphi_i^{-1})$$

$\varphi_i : \mathrm{dom}\,\varphi_i \xrightarrow{\sim} \mathrm{im}\,\varphi_i$ bijections.

$\mathrm{graph}\,\varphi = \{(\varphi x, x) \mid x \in \mathrm{dom}\, x\}$



$$\to \cdots \to \bigoplus_{i=1}^{n} \mathbb{Z}_G \xrightarrow{\partial_0} \mathbb{Z}_G \xrightarrow{\varepsilon} \ell^{\infty}(X) \to 0$$

$\varepsilon(1_{\Delta_X}) = 1_X$

$\partial_0(1_{\Delta_X}[i]) = 1_{\mathrm{graph}\,\varphi_i} - 1_{\Delta_{\mathrm{im}\,\varphi_i}}$

- If G is acyclic :

$\mathrm{im}\,\partial_0 \simeq \bigoplus_{i=1}^{n} \mathbb{Z}_G \cdot 1_{\Delta_{\mathrm{im}\,\varphi_i}}$ is projective.

$\rightsquigarrow \mathrm{cd}_{\mathbb{Z}}(G) \leq 1$.

<u>Prop</u> $\mathrm{cd}_{\mathbb{Z}}(G) = 0 \Leftrightarrow G$ is uniformly finite.

prf: $\mathrm{cd}_{\mathbb{Z}}(G) = 0 \Leftrightarrow \ell^\infty(X)$ is projective.

$$\Leftrightarrow \quad \mathbb{Z}G \underset{\exists \sigma}{\overset{\varepsilon}{\rightleftarrows}} \ell^\infty(X) \qquad \varepsilon \circ \sigma = \mathrm{id}_{\ell^\infty(X)}$$

$a := \sigma(1_X)$ satisfies

(1) $\varepsilon(a) = 1_X$

i.e., $\sum_{z \in X} a(x, z) = 1$

(2) a is left-inv.

i.e., $a(x\ z) = a(y, z)$ if $x, y \in [z]_G$.

$\rightsquigarrow$ For each equiv. class,

$\exists z$ s.t. $a(x, z) = a(y, z) \neq 0 \quad \forall x, y \in [z]_G$

$\rightsquigarrow$ $\mathrm{diam}([z]_G)$ is uniformly bounded.

$\square$.

Now consider $H^n(G, \mathbb{Z}G)$:

$$
\begin{tikzcd}
\cdots & \mathrm{Hom}_{\mathbb{Z}G}\left(\bigoplus_{i=1}^{n} \mathbb{Z}G, \mathbb{Z}G\right) \arrow[l] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}G) \arrow[l, "\partial_0^*"'] \arrow[d, phantom, "\cong" rotate=90] \\
& \bigoplus_{i=1}^{n} \mathbb{Z}G & \mathbb{Z}G
\end{tikzcd}
$$

is a seq of right $\mathbb{Z}G$-modules

$\rightsquigarrow$ $H^n(G, \mathbb{Z}G)$ are right $\mathbb{Z}G$-modules.

<u>Prop</u>. (1) $cd_{\mathbb{Z}}(G) \leq 1$

$\Rightarrow H^1(G, \mathbb{Z}G)$ is a f.g. right $\mathbb{Z}G$-module.

(2) $cd_{\mathbb{Z}}(G) \leq 1$ & $H^1(G, \mathbb{Z}G) = 0$

$\Rightarrow cd_{\mathbb{Z}}(G) = 0$.

- Decomposition of Borel graphs

<u>Thm 2</u> (I.)

(X, G) a Borel graph w/ UBD.

Suppose that $H^1(G, \mathbb{Z}_G)$ is a f.g. right $\mathbb{Z}_G$-module.

Then $\exists (Y, G')$ a Borel graph w/ UBD

- $(X, G) \hookrightarrow (Y, G')$ Lipschitzly

  (i.e. $f : X \to Y$ Borel injective
  $\exists L \geq 1 \ \forall x, x' \in X$,
  $L^{-1} d_G(x, x') \leq d_{G'}(f(x), f(x')) \leq L\, d_G(x, x')$.

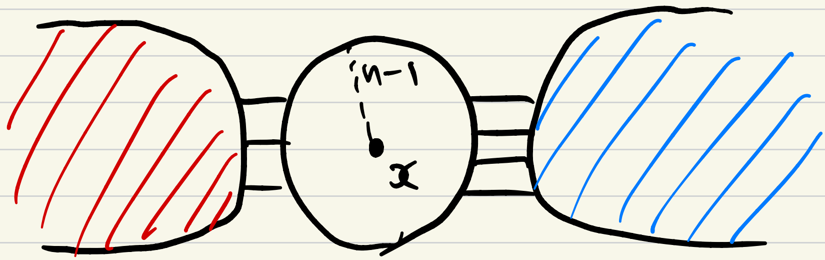
- $G' = T * K$ w/ T and K Borel subgraphs
- T is acyclic.
- $H^1(K, \mathbb{Z}_K) = 0$.

- What is $H^1(G, \mathbb{Z}_G)$?

$W_G^n :=$ the set of $f \in \ell^\infty(E_G)$

s.t. $\forall x \in X\ \forall (y, z) \in G_x$

$y, z \notin B_{n-1}(x) \Rightarrow f(y, x) = f(z, x)$



$W_G := \bigcup_{n=0}^{\infty} W_G^n$ is a right $\mathbb{Z}_G$-module :

<u>Prop</u>. $H^1(G, \mathbb{Z}_G) \simeq W_G / W_G^0 + \mathbb{Z}_G$.

Def. A cut of G is a subset $C \subset [x]_G$ of an E_G-class s.t. $\partial_v C$ is bounded.

Def. $\mathcal{C}$ a family of cuts of G.
$H^1(G, \mathbb{Z}_G)$ is generated by $\mathcal{C}$
$\Leftrightarrow$ W_G is generated by
$$\{ f \in W_G \mid \forall x \in X \ \exists C \in \mathcal{C}, f(\cdot, x) = 1_C \} \cup (\mathbb{Z}_G + W_G^0)$$

Prop. (1) $H^1(G, \mathbb{Z}_G)$ is f.g.
$\Leftrightarrow$ it is generated by a family of cuts w/ uniformly bounded boundaries.
(2) $H^1(G, \mathbb{Z}_G) = 0$
$\Leftrightarrow$ it is generated by the empty family of cuts.

- Proof of Thm 2

<u>Strategy</u> : Suppose $H^1(G, \mathbb{Z}_G)$ is f.g.

1. $\exists n$, $\mathcal{C} := \{ C \mid \mathrm{diam}(C) \leq n \}$ generates $H^1(G, \mathbb{Z}_G)$.

2. $\mathcal{C} = \bigsqcup_{k=1}^{m} \mathcal{C}_k$, where $\mathcal{C}_k$ is a Borel family

s.t. • complement-closed

• nested

i.e. $\forall C, D \in \mathcal{C}_k$, either

$C \cap D$, $\bar{C} \cap D$, $C \cap \bar{D}$ or $\bar{C} \cap \bar{D}$ is empty.

3. $G \underset{\mathcal{C}_1}{\hookrightarrow} T_1 * G_1 \underset{\mathcal{C}_2}{\hookrightarrow} T_1 * T_2 * G_2 \hookrightarrow$

$\cdots \underset{\mathcal{C}_m}{\hookrightarrow} T_1 * \cdots * T_m * G_m$

For simplicity, assume $\mathcal{C} = \mathcal{C}_1$.

Fix $(x, y) \in G$.

$\rightsquigarrow$ $\{ C \in \mathcal{C} \mid x \in C \not\ni y \}$ is finite

and linearly ordered:

$x \in C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_\ell \not\ni y$

Do this operation for all edge

$\rightsquigarrow$ Get a Borel graph (Y, G').

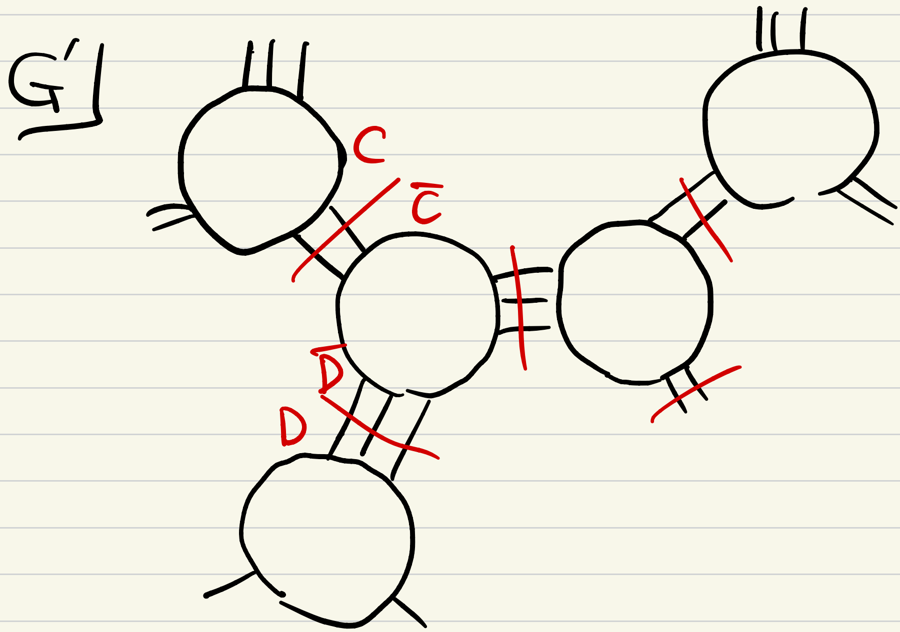
Then • $(X, G) \hookrightarrow (Y, G')$ is Lipschitz.

• $\mathcal{C}$ is regarded as a Borel nested family of cuts of G.

• $\mathcal{C}$ generates $H^1(G', \mathbb{Z}_{G'})$.

Made with Goodnotes

Now each edge of G' is separated by at most one pair of opposite cuts.



Fix a pair $(C, \bar{C})$

1. Select one edge separated by this.

2. Add new edges

$$(\partial_{iv} C \times \partial_{iv} C) \cup (\partial_{iv} \bar{C} \cup \partial_{iv} \bar{C})$$

We get $G'' = T * K \overset{\text{Lip}}{\sim} G'$

T : the edges separated by $\hat{C}$

K : ——— not separated by $\hat{C}$.

$\rightsquigarrow$ • T is acyclic

• $H^1(K, \mathbb{Z}_K)$ is generated by the empty cuts.

$\square$